Noninvertible Solitonic Symmetry

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Based on arXiv:2210.13780, 2307.00939

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Conservation law of topological solitons is NOT fully characterized by homotopies $\pi_*(\mathcal{M})$.

Its algebraic structure can be more complicated $\Longrightarrow$ Noninvertible Solitonic Symmetry

Outline

1. Introduction

2. 4d $\mathbb{C}P^1$ σ-model & Hopfion

3. Mathematical structure

4. Summary

When (usual) symmetry is spontaneously broken

$$G \xrightarrow{SSB} H,$$

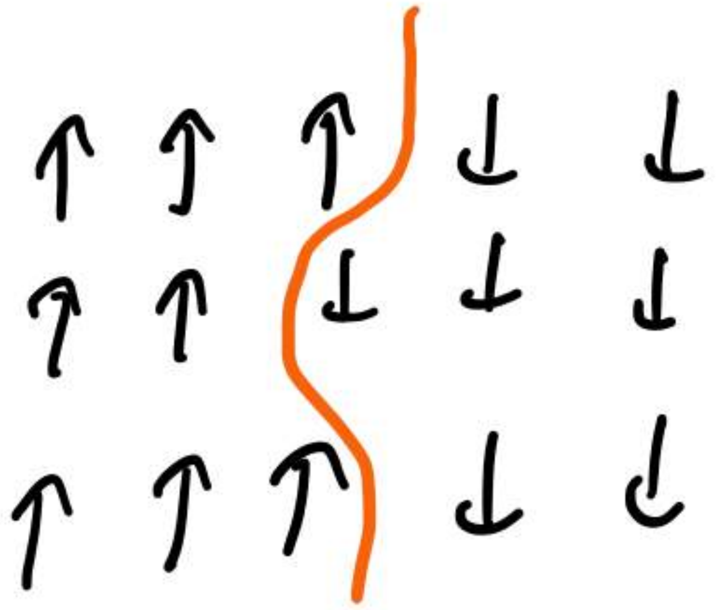
the low-energy effective theory is given by a non-linear σ-model:

$$\sigma : \text{Spacetime} \longrightarrow G/H$$

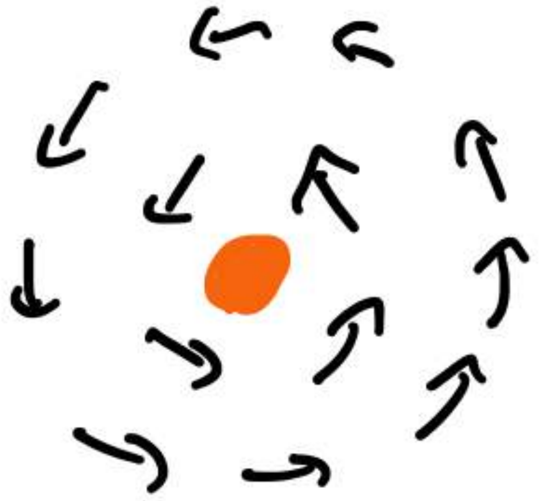
$$Z = \int \mathcal{D}\sigma \, \exp\left(-\frac{1}{g^2} \int |d\sigma|^2 + (\cdots\cdots) \right).$$

Small fluctuations of σ : Nambu-Goldstone modes

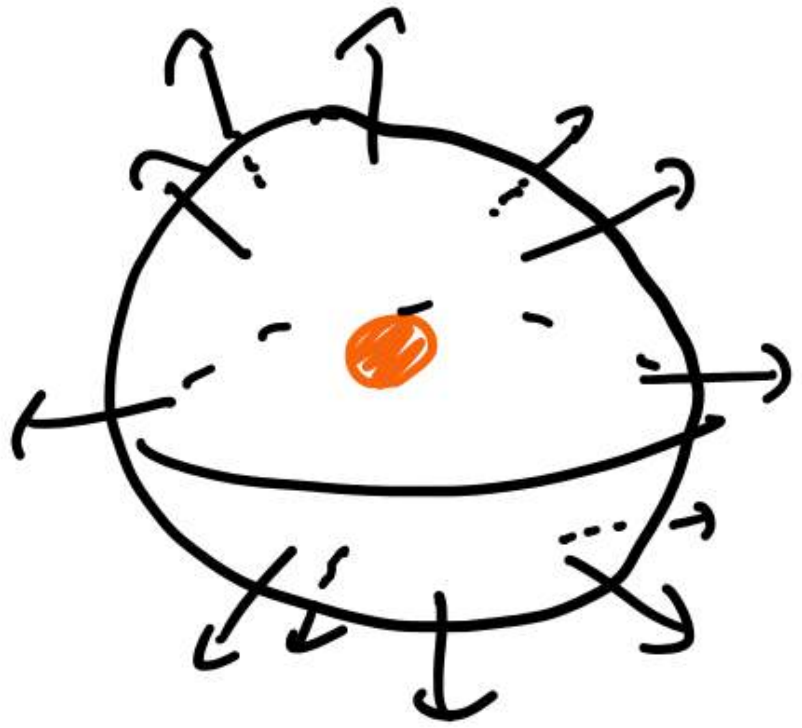
Large fluctuation : Topological Solitons



domain wall



vortex

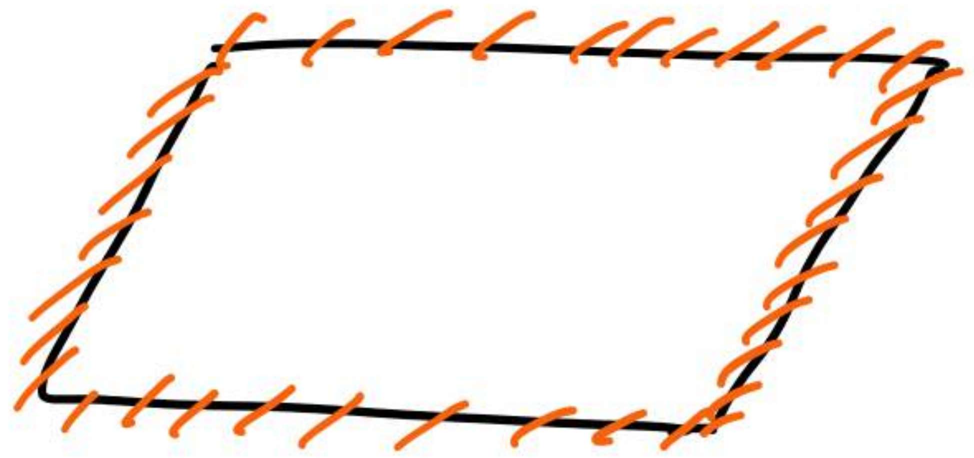


hedgehog

Topological Stability and Homotopies

(Mermin '79 Rev. Mod. Phys.)

Spacetime



$\xrightarrow{\sigma} \mathcal{M} (= G/H)$.

To find finite action/energy (density) $\int |d\sigma|^2 < \infty$, its convenient to identify ∞'s of $\mathbb{R}^n$:

$$\mathbb{R}^n \cup \{\infty\} \simeq S^n \xrightarrow{\sigma} \mathcal{M}.$$

$\Rightarrow$ Topoligcal solitons are classified by homotopies of the target space

$$\pi_n(\mathcal{M}).$$

Recall that

Conservation Law $\Longleftrightarrow$ Symmetry,

we should be able to understand this topological conservation law as the symmetry of the σ-model.

Is this always true?

$\boxed{\text{Conventional Wisdom: Solitonic Sym.} \cong \mathrm{Hom}(\pi_n(\mathcal{M}), U(1)).}$

4d $\mathbb{C}P^1$ σ-model

Assume some $(3+1)$ d quantum systems have SSB

$$SU(2) \xrightarrow{SSB} U(1).$$

The target space of the nonlinear σ-model becomes

$$\mathbb{C}P^1 \simeq SU(2)/U(1).$$

Lagrangian :

$$\begin{cases} \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & : \mathbb{C}^2\text{-valued scalar field with } |\vec{z}|^2 = 1. \\ a = a_\mu dx^\mu & : \text{(auxiliary) } U(1) \text{ gauge field} \end{cases}$$

$$\mathcal{L} = \frac{1}{g^2} \left| (\partial_\mu + i a_\mu) \vec{z} \right|^2.$$

This $U(1)$ gauge field a is auxiliary because its EoM can be solved as

$$a = i \vec{z}^\dagger \cdot d\vec{z}$$

Homotopy of $\mathbb{C}P^1$ $(\simeq S^2)$:

$$\pi_1(\mathbb{C}P^1) \simeq 0 \quad , \quad \underline{\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}} \quad , \quad \underline{\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}}$$

"magnetic skyrmion" "Hopfion"

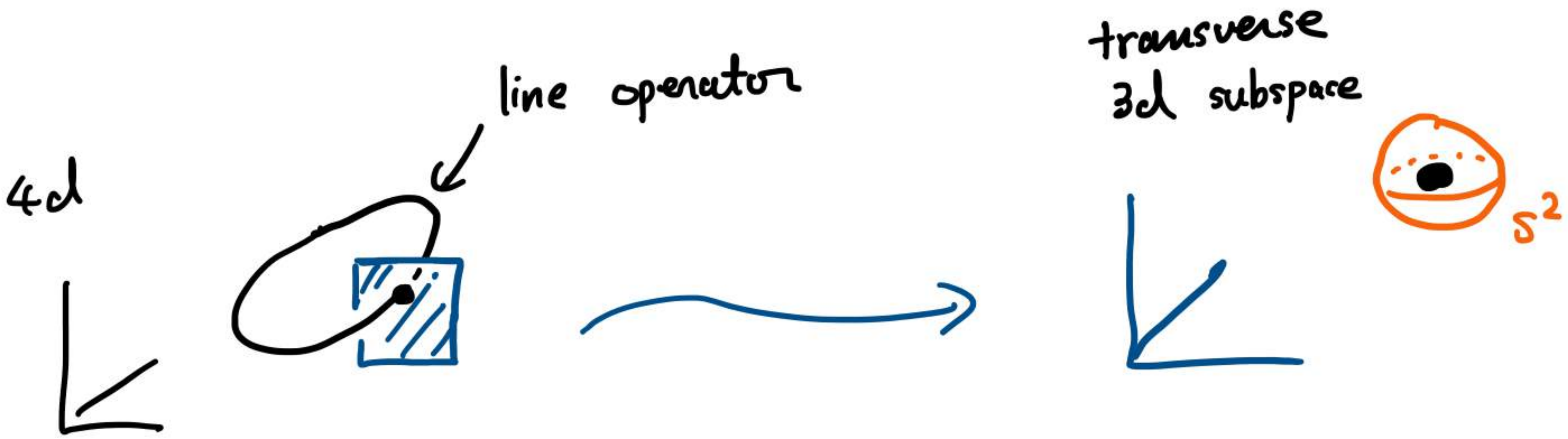
"monopole"

Vortex Soliton $\pi_2(\mathbb{CP}^1)$

$\pi_2(\mathbb{CP}^1) \simeq \mathbb{Z}$ has the Noether current

$$j = \frac{1}{2\pi} da,$$

and it gives $U(1)$ 1-form symmetry.



$$\underbrace{\int_{S^2} \frac{da}{2\pi} \in \mathbb{Z}}_{\text{Dirac quantization}} \simeq \pi_2(\mathbb{CP}^1)$$

We denote this line operator as $V_n(L)$ with $n = \int_{S^2} \frac{da}{2\pi}$.

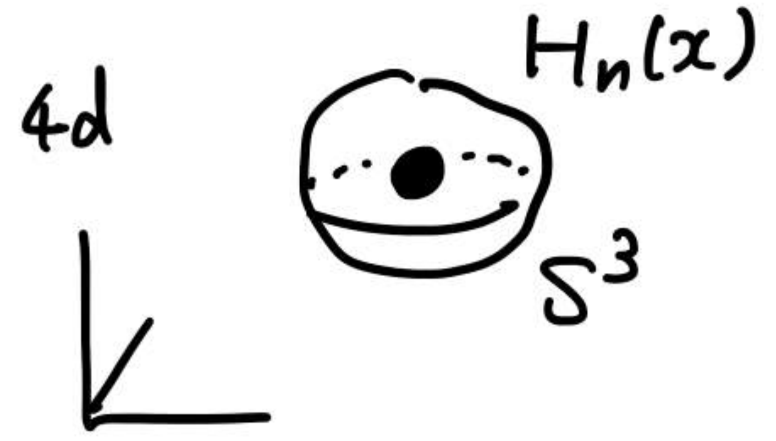
(We'll see that there is a finer classification for the vortex operators $V_n(L)$)

Hopfion $\pi_3(\mathbb{C}P^1)$

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ follows from the Hopf fibration $S^1 \to S^3 \to S^2$, and the corresponding solitons are known as "Hopfion" (or "Hopf soliton").

$\pi_3(\mathbb{C}P^1)$ is measured by the Hopf number [Wilczek, Zee '83]

$$\frac{1}{4\pi}\int_{S^3} a\,da \in \pi\mathbb{Z}$$



(* For general $U(1)$ gauge fields, the Chern-Simons form $\int \frac{1}{4\pi} a\,da$ can take arbitrary numbers.
Here, since a is an "auxiliary" field $(a = i\vec{z}^\dagger d\vec{z})$,
$d\left(\frac{1}{4\pi} a\,da\right) = \frac{1}{4\pi}(da)^2 = 0$. $\Rightarrow \int \frac{1}{4\pi} a\,da$ becomes quantized.)

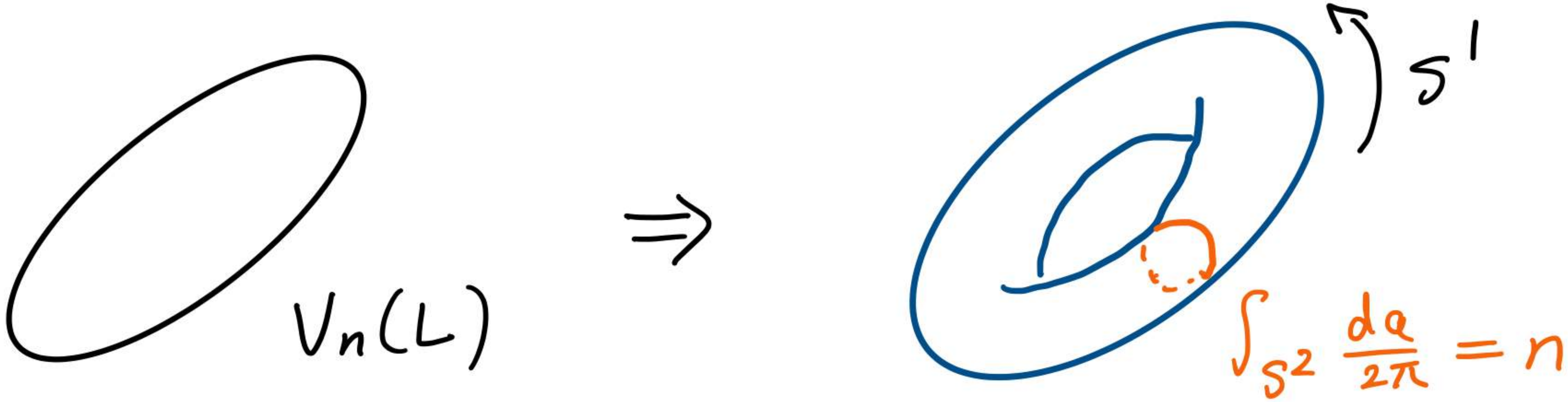
Unlike the case of $\pi_2(\mathbb{C}P^1)$, however, the integrand

$$j_{\text{Hopf}} := \frac{1}{4\pi^2}\, a\,da$$

is not gauge invariant.

$$\boxed{U(1) \text{ without Noether current ?}}$$

More on Vortex Operators $V_{n,k}(L)$



Let's classify the $\mathbb{CP}^1$ configurations

$$S^2\times S^1 \xrightarrow{\sigma} \mathbb{CP}^1$$

up to homotopy for a given monopole charge $\int_{S^2}\frac{da}{2\pi}=n$.

In particular, what is the possible value of

$$\int_{S^2\times S^1}\frac{a\,da}{4\pi^2}\ (=: k)\quad ? \qquad \Rightarrow\ V_{n,k}(L).$$

$\mathbb{Z}_{2n}$ classification : $V_{n,k+2n}(L) \cong V_{n,k}(L)$

Let's try to evaluate the Hopfion number on $S^2 \times S^1$:

$$k = \int_{S^2 \times S^1} \frac{ada}{4\pi^2}.$$

Under the large $U(1)$ gauge transformation $a \mapsto a + \varepsilon^{(1)}$ along S^1,

$$\int_{S^2\times S^1} \frac{ada}{4\pi^2} \longmapsto \int_{S^2\times S^1} \frac{ada}{4\pi^2} + \frac{1}{\pi} \underbrace{\int_{S^1} \varepsilon^{(1)}}_{\in 2\pi\mathbb{Z}} \underbrace{\int_{S^2} \frac{da}{2\pi}}_{=n}$$

$$= \int_{S^2\times S^1} \frac{ada}{4\pi^2} + 2n\mathbb{Z}.$$

$$\Rightarrow \int_{S^2\times S^1} \frac{ada}{4\pi^2} = k \text{ is well-defined only in } \mathbb{Z}_{2n}, \text{ i.e. } k \sim k+2n.$$

[cf. Pontrjagin '41]

Is the Hopfion symmetry $U(1)$ or $\mathbb{Z}_2$?

Let's evaluate correlation functions in a compact spacetime.

$$\left\langle H_{k_1}(x_1) \quad H_{k_2}(x_2) \quad H_{k_3}(x_3) \right\rangle \neq 0 \implies k_1 + k_2 + k_3 + \cdots = 0.$$

$U(1)$ conservation law

$$\left\langle H_{k_1}(x_1) \quad H_{k_2}(x_2) \quad V_{n,k}(L) \right\rangle \neq 0 \implies k_1 + k_2 + \cdots + k = 0 \mod 2n.$$

$\mathbb{Z}_{2n}$ conservation law.

$$\begin{cases} \text{Without } V(L), \text{ the Hopfion number conserves as if there is a } U(1) \text{ symmetry.} \\ \text{With } V(L), \text{ the conservation law reduces to that of } \mathbb{Z}_2. \end{cases}$$

Which is the symmetry group? Or, is it something else?

Generalized Symmetry in QFTs

$$\boxed{\text{Generalized Symmetry} \;=\; \text{Topological Operators}}$$

For continuous symmetry,

$Q(M_{d-1}) = \int_{M_{d-1}} \mathcal{J}$ is invariant under any topological continuous deformation of M_{d-1}.

In conventional symmetry, those topological operators obey group structures.

However, this turns out to be too restrictive to explore QFTs.

Non-invertible symmetry (or Categorical symmetry)

$$a \;\; b \;=\; \sum_c N^c_{ab}(M_{d-1}) \;\; c$$

Fusion rule of symmetry defects can be quite general.

(2d CFTs : Verlinde '88, Bhadwaj, Tachikawa '17, Thorngren, Wang '19, ...

Higher dims (Since '21) : Nguyen, YT, Ünsal ; Heidenreich, McNamara, Reece, Rudelius, Valenzuela; Koide, Nagoya, Yamaguchi ; Choi, Cordova, Hsin, Lam, Shao ; Kaidi, Ohmori, Zheng ; ...)

Topological operators and TQFTs

One of useful methods to find "unconventional" sym.: (cf. Choi, Lam, Shao '22; Cordova, Ohmori '22)

1. Prepare a TQFT
2. Put it on a submanifold with a topological coupling to dynamical fields.

(As every ingredient is topological, this operator is manifestly topological. We should check if it acts nontrivially to local operators.)

In our case,

1. We prepare the level-N $U(1)$ CS theory $\int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int bdb}$

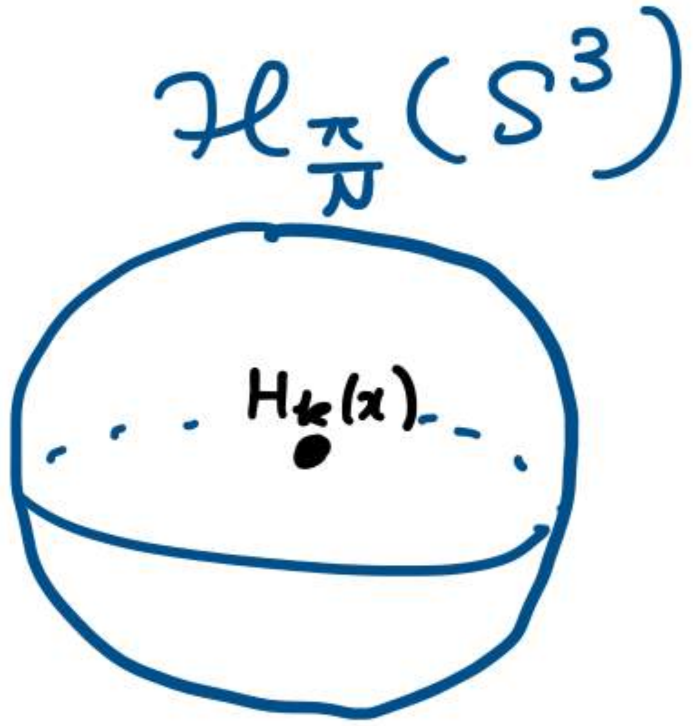
2. The Hopfion symmetry operator is then defined as

$$\boxed{\mathcal{H}_{\frac{\pi}{N}}(M_3, da) = \int \mathcal{D}b \, \exp\left(i\frac{N}{4\pi}\int_{M_3} b \wedge db + i\frac{1}{2\pi}\int_{M_3} b \wedge da \right).}$$

Let's check how this operator acts on $\underbrace{H_k(x)}_{\text{Hopfion op}}$ and $\underbrace{V_{n,k}(L)}_{\text{vortex op.}}$.

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $H_k(x)$

To evaluate the action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$, we can set $M_3 = S^3$ that surrounds x.



$$\mathcal{H}_{\frac{\pi}{N}}(S^3) = \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

$$= \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int_{S^3}(b+\frac{a}{N})\,d(b+\frac{a}{N})} \cdot e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}$$

On S^3, $U(1)$ bundle is trivial.

$$\propto e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}.$$

This shows that

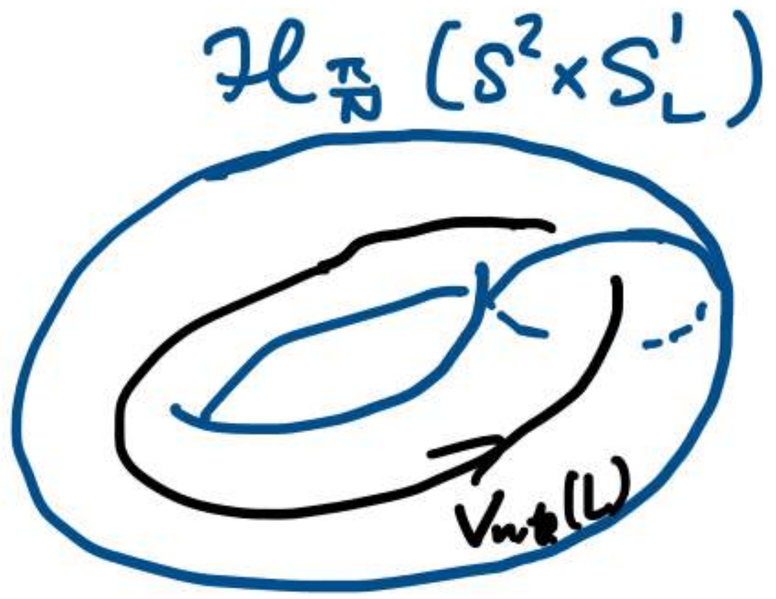
$$\left\langle \mathcal{H}_{\frac{\pi}{N}}(S^3)\, H_k(x) \cdots \right\rangle \propto e^{-i\frac{\pi}{N}k} \left\langle H_k(x) \cdots \right\rangle.$$

and $\mathcal{H}_{\frac{\pi}{N}}(S^3)$ detects the Hopfion charge of $H_k(x)$.

Since N can be arbitrary, $\{\mathcal{H}_{\frac{\pi}{N}}(S^3)\}_{N\geq 1}$ determines k as an integer.

$\Rightarrow$ Recovery of $U(1)$-like selection rule

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $V_{n,k}(L)$



Next, we set $M_3 = S^2 \times S^1_L$ to evaluate its action on $V_{n,k}(L)$.

$$\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1) = \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

$$= \sum_m (\cdots) \underbrace{\int_0^{2\pi} d\beta \, e^{i\overset{\int_{S^1} b}{\beta}\left(N\overset{\int_{S^2}\frac{db}{2\pi}}{m} + \overset{\int_{S^2}\frac{da}{2\pi}}{n}\right)}}_{=0 \text{ unless } Nm+n=0.}$$

U(1) bundle over $S^2\times S^1$ can be nontrivial.

$$= \begin{cases} e^{-i\frac{\pi}{n}k} & \text{for } N=n. \\ 0 & \text{if } N \text{ is not a divisor of } n. \end{cases}$$

When the symmetry looks to be reduced to $\mathbb{Z}_{2n}$ by the presence of vortex operators, $\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1)$ acts nontrivially only if it fits the periodicity of $\mathbb{Z}_{2n}$.

Moreover, $\mathcal{H}_{\frac{\pi}{n}}(S^2\times S^1)$ captures the Hopfion charge of $V_{n,k}(L)$ in mod $2n$.

For 4d $\mathbb{C}P^1$ σ-model,

the Hopfion symmetry associated with $\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ is neither $U(1)$ nor $\mathbb{Z}_2$.

The correct symmetry generator is given by

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \, e^{i \frac{N}{4\pi} \int_{M_3} b \wedge db + i \frac{1}{2\pi} \int_{M_3} b \wedge da},$$

and the fusion rule is controlled by those of 3d TQFTs.

There is an invertible $\mathbb{Z}_2$ subgroup, generated by

$$\mathcal{H}_{\pi}(M_3) = e^{i\pi \int_{M_3} \frac{a\, da}{4\pi^2}} \in \mathbb{Z}_2 \left(\simeq \tilde{\Omega}_3^{Spin}(\mathbb{C}P^1) \right).$$

Mathematical Formulation of Solitonic Symmetry

Let us give a (tentative) proposal for general structure of solitonic symmetry.

$$Z = \int_{M \to Y} \mathcal{D}\sigma \; e^{-S[\sigma]}$$

- Here, Y can be infinite-dim. space formally, so that σ contains higher-gauge fields
- We assume that $|\pi_\bullet(Y)| < \infty$, and $\pi_k(Y) = 0$ for sufficiently large k.

Solitonic symmetries should be generated by topological functionals of σ.

Requiring locality, it would be natural to argue that

$$\boxed{J[M,\sigma] = \text{Partition function of fully-extended } Y\text{-enriched TQFT.}}$$

Fully-extended TQFTs are mathematical formulations of TQFTs extending Atiyah's axiom to include locality. [Baez, Dolan '95, Lurie '09]

$$Z : \text{Bordism category} \xrightarrow[\text{monoidal}]{\text{symmetric}} \text{Vec. category} \qquad (\text{Atiyah})$$

$$\Downarrow$$

$$Z : \text{Bord. } (\infty, n)\text{-cat.} \longrightarrow (\infty, n)\text{-cat.} \qquad (\text{fully-extended version})$$

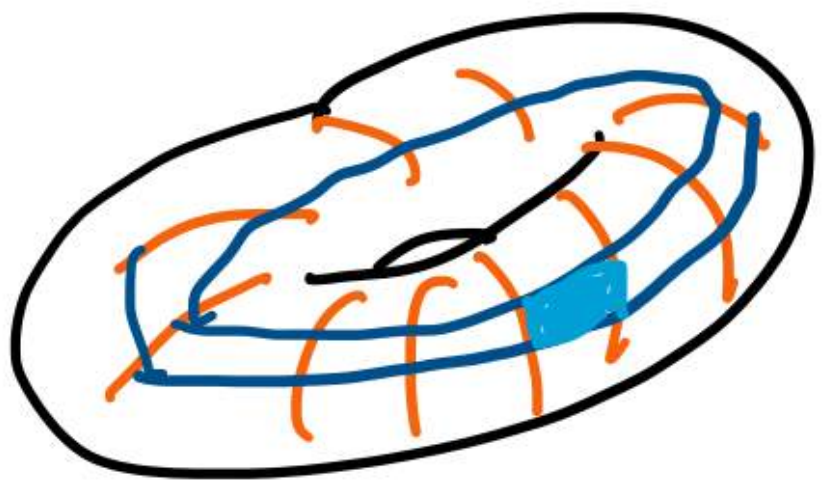
for $q \leq n$, q-morphisms are q-dim. manifolds w/ corners.

<u>Cobordism hypothesis</u>

Fully-extended TQFT $Z : \text{Bord}_n \longrightarrow \mathcal{C}$

$$\overset{1:1}{\Longleftrightarrow} \quad Z(*) \in \mathrm{Ob}(\mathcal{C})$$

(Idea)



Any manifolds can be constructed by gluing an open disk .

If we evaluate $Z(\;) \in \mathrm{Ob}(\mathcal{C})$, it determines the whole data of local TQFT.

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Let us define Y-enriched TQFTs following this idea:

<u>Def</u> n-dim. fully-extended Y-enriched TQFTs are given by

$$Z : \underbrace{\mathrm{Bord}_n(Y)}_{\text{Y-enriched } (\infty,n)\text{-bord.}} \longrightarrow \underbrace{\Sigma^{n-1}{}_{(s)}\mathrm{Vec}}_{\text{target } (\infty,n)\text{-category}} .$$

proposed by Gaiotto, Johnson-Freyd '19.

According to cobordism hypothesis, the equivalent data can be obtained after point evaluation:

$$\underline{Z} : Y \to \Sigma^{n-1}{}_{(s)}\mathrm{Vec}$$

We call this functor $\underline{Z}$ an n-representation associated with Z.

(Here, Y is identified with its homotopy ∞-type, i.e. ∞-groupoid.)

Thus, topological functionals of Y turn out to be described by these functors,

$$\boxed{{}_{(s)}\mathrm{Rep}^{\bullet}(Y) := \mathrm{Func}(Y, \Sigma^{\bullet-1}{}_{(s)}\mathrm{Vec})}$$

We call it solitonic cohomology. As an invertible part,

$$\mathrm{Rep}^{\bullet}(Y) \supset H^{\bullet}(Y; \mathbb{C}^{\times}).$$

Summary

- Topological conservation law for solitons is not fully characterized by Homotopies.

- 4d $\mathbb{C}P^1$ σ-model is carefully examined.

$$\begin{cases} \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z} & \Rightarrow \; U(1) \text{ 1-form symmetry for vortex.} \\ \pi_3(\mathbb{C}P^1) \simeq \mathbb{Z} & \cancel{\Rightarrow} \; U(1) \text{ symmetry for Hopfion.} \end{cases}$$

$$\left\langle H_{k_1}(x_1) \; H_{k_2}(x_2) \; V_{n,k}(L) \right\rangle \neq 0 \;\Rightarrow\; k_1 + k_2 + \cdots + k = 0 \mod 2n .$$

- The symmetry generators are given by 3d TQFT partition functions

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \; \exp\left(i \int_{M_3} \frac{N}{4\pi} b \, db + i \int_{M_3} \frac{1}{2\pi} b \, da \right).$$

$\Rightarrow$ Noninvertible Solitonic Symmetry

- We introduce the solitonic cohomology $\mathrm{Rep}^{\bullet}(Y) = \mathrm{Func.}(Y, \Sigma^{\bullet - 1}\mathrm{Vec})$ as the mathematical formulation of solitonic symmetry.